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Can the Wigner function be determined by properties for translation and parity transformation on lattice 'phase space'?

Minoru Horibe, Takaaki Hashimoto and Akihisa Hayashi

Department of Applied Physics, Fukui University, Fukui 910, Japan

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Abstract

We show that the Fano operator for a quantum system confined to a line $\mathbb{R}$ is uniquely determined by assuming reasonable behaviour under translation and parity transformation on phase space. In contrast, for a system on a lattice the same procedure does not work.

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1. Introduction

The expectation values for a mixed state with the density matrix $\hat{\rho}$ are expressed as the averages over phase-space quasiprobability distribution $W(q, p)$ defined by

$$W(q,p) = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} \mathrm{d}r \left[\mathrm{e}^{-\mathrm{i}pr/\hbar} \left\langle q + \frac{r}{2}, \hat{Q} \middle| \hat{\rho} \middle| q - \frac{r}{2}, \hat{Q} \right\rangle \right] \tag{1}$$

which is well known as the Wigner function (Wigner 1932), where $\left|q \pm \frac{r}{2}, \hat{Q}\right\rangle$ is the eigenfunction with eigenvalue $q \pm \frac{r}{2}$ for the coordinate operator. We can check that this function satisfies the following conditions:

(A) We can obtain the marginal distribution along the coordinate and the momentum axes,

$$\int_{-\infty}^{\infty} W(q,p)\,\mathrm{d}p = \langle q, \hat{Q}|\hat{\rho}|q, \hat{Q}\rangle$$

$$\int_{-\infty}^{\infty} W(q,p)\,\mathrm{d}q = \langle p, \hat{P}|\hat{\rho}|p, \hat{P}\rangle.$$

(B) The Wigner function is a real-valued function,

$$W^*(q,p) = W(q,p).$$

(C) The Wigner function includes the same information as the density matrix.

Here, $|q, \hat{Q}\rangle$ and $|p, \hat{P}\rangle$ are eigenfunctions for the coordinate operator $\hat{Q}$ and the momentum operator $\hat{P}$, respectively,

$$\hat{Q}|q, \hat{Q}\rangle = q|q, \hat{Q}\rangle \qquad \hat{P}|p, \hat{P}\rangle = p|p, \hat{P}\rangle.$$

Conversely, it has been pointed out that these conditions do not determine the Wigner function uniquely (for example, Krügert and Poffyn 1976, Wigner 1979, O'Connel and Wigner 1981, Tatarskii 1983, Takami *et al* 2001). For this reason, Bertrand and Bertrand (1987) and Leonhardt (1997) impose an additional condition which gives the connection between rotations of quantum variables $(\hat{Q}, \hat{P})$ and of the point (q, p) in phase space on which the Wigner function is defined,

$$\langle q, \hat{Q}|R_\theta \hat{\rho} R_\theta^{-1}|q, \hat{Q}\rangle = \int_\infty^\infty W(q\cos\theta + p\sin\theta, -q\sin\theta + p\sin\theta)\,\mathrm{d}p$$

where R_θ is the unitary operator for rotation of quantum variables $(\hat{Q}, \hat{P})$,

$$R_\theta \hat{Q} R_\theta^{-1} = \hat{Q}\cos\theta - \hat{P}\sin\theta \qquad R_\theta \hat{P} R_\theta^{-1} = \hat{P}\cos\theta + \hat{Q}\sin\theta.$$

And they show that there is only one solution satisfying it for a quantum system confined to a line $\mathbb{R}$. This condition makes it possible for us not only to determine the Wigner function uniquely, but also to infer it from the quantity $\langle q, \hat{Q}|R_\theta \hat{\rho} R_\theta^{-1}|q, \hat{Q}\rangle$ obtained by observation (Leonhardt 1996).

In the previous paper (Horibe *et al* 2002), we rewrote this condition using the Fano operator $\hat{\Delta}(q, p)$ (Fano 1957) defined by

$$W(q, p) = \mathrm{Tr}[\hat{\Delta}(q, p)\hat{\rho}] \tag{2}$$

$$\hat{\rho} = \frac{1}{2\pi\hbar}\int_\infty^\infty W(q, p)\hat{\Delta}^\dagger(q, p)\,\mathrm{d}q\,\mathrm{d}p. \tag{3}$$

Namely, we assumed that

$$R_\theta \hat{\Delta}(q, p) R_\theta^{-1} = \hat{\Delta}(q\cos\theta + p\sin\theta, -q\sin\theta + p\sin\theta) \tag{4}$$

and showed that there is only one solution satisfying this condition. For the lattice 'phase space' with N^2 sites, we could discuss it in the same manner and found the unique solution, which is equivalent to that given by Cohendet *et al* (1988), for the case where N is odd, but no solution for the case where N is even. In the strict sense, the lattice space with N^2 sites may not be a phase space. However we used and will use the word 'phase space' for this space in this paper since, for the system whose Hilbert space is the N-dimensional one, this space corresponds to the phase space of the usual dynamical system.

Naively, we are interested in whether the Wigner function is determined uniquely under the assumption of properties of simpler transformation than rotation. In this paper, we assume the behaviour of the Fano operator $\hat{\Delta}(q, p)$ in equation (2) under the translation and parity transformation and try to determine the Fano operator. For a quantum system confined to a line $\mathbb{R}$, we can find only one Fano operator which satisfies the conditions corresponding to the above ones (A)–(C) and new conditions. But, for systems on a lattice 'phase space', we cannot determine it uniquely.

2. The Wigner function for a one-dimensional system

In this section, we study the Fano operator $\hat{\Delta}(q, p)$ defined by equations (2) and (3). We shall give a derivation that is not the shortest, but which we shall modify in section 3. In terms of the Fano operator, we can rewrite conditions (A)–(C) in the preceding section,

$$\int_{-\infty}^{\infty} \hat{\Delta}(q, p)\,\mathrm{d}p = |q, \hat{Q}\rangle\langle q, \hat{Q}| \tag{5}$$

$$\int_{-\infty}^{\infty} \hat{\Delta}(q, p)\,\mathrm{d}q = |p, \hat{P}\rangle\langle p, \hat{P}| \tag{6}$$

$$\hat{\Delta}^{\dagger}(q, p) = \hat{\Delta}(q, p) \tag{7}$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (\hat{\Delta}^{\dagger})_{q_1 q_2}(q, p)(\hat{\Delta})_{q_3 q_4}(q, p)\,\mathrm{d}p\,\mathrm{d}q = \frac{1}{2\pi\hbar}\delta(q_1 - q_4)\delta(q_3 - q_2) \tag{8}$$

where $|q, \hat{Q}\rangle$ and $|p, \hat{P}\rangle$ are eigenfunctions of the coordinate operator $\hat{Q}$ and the momentum operator $\hat{P}$ with eigenvalues q and p, respectively, as stated previously and $\Delta_{q_1 q_2}(q, p) = \langle q_1, \hat{Q}|\hat{\Delta}(q, p)|q_2, \hat{Q}\rangle$ is a matrix element of the Fano operator between eigenfunctions of the operator $\hat{Q}$ for eigenvalues q_1 and q_2. When we expand the Fano operator in terms of the complete set $\mathrm{e}^{\mathrm{i}\mathcal{Q}\hat{P}/\hbar}\,\mathrm{e}^{-\mathrm{i}\mathcal{P}\hat{Q}/\hbar}$,

$$\hat{\Delta}(q, p) = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} \mathrm{d}\mathcal{Q}\,\mathrm{d}\mathcal{P}\, a(q, p; \mathcal{Q}, \mathcal{P})\,\mathrm{e}^{\mathrm{i}\mathcal{Q}\hat{P}/\hbar}\,\mathrm{e}^{-\mathrm{i}\mathcal{P}\hat{Q}/\hbar} \tag{9}$$

the coefficients $a(q, p; \mathcal{Q}, \mathcal{P})$ should satisfy the conditions

$$\int_{-\infty}^{\infty} a(q, p; \mathcal{Q}, \mathcal{P})\,\mathrm{d}p = \delta(\mathcal{Q})\,\mathrm{e}^{\mathrm{i}q\mathcal{P}/\hbar}$$

$$\int_{-\infty}^{\infty} a(q, p; \mathcal{Q}, \mathcal{P})\,\mathrm{d}q = \delta(\mathcal{P})\,\mathrm{e}^{-\mathrm{i}p\mathcal{Q}/\hbar}$$

$$a^{*}(q, p; \mathcal{Q}, \mathcal{P}) = \mathrm{e}^{-\mathrm{i}\mathcal{Q}\mathcal{P}/\hbar} a(q, p; \mathcal{Q}, \mathcal{P})$$

$$\int_{\infty}^{\infty} a^{*}(q, p : \mathcal{Q}, \mathcal{P}) a(q, p : \mathcal{Q}', \mathcal{P}')\,\mathrm{d}q\,\mathrm{d}p = \delta(\mathcal{Q} - \mathcal{Q}')\delta(\mathcal{P} - \mathcal{P}')$$

because of conditions (5)–(8). Using the Fourier transformed coefficients $\tilde{a}(s, t; \mathcal{Q}, \mathcal{P})$

$$\tilde{a}(s, t; \mathcal{Q}, \mathcal{P}) = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} a(q, p; \mathcal{Q}, \mathcal{P})\,\mathrm{e}^{-\mathrm{i}(qs - pt)/\hbar}\,\mathrm{d}q\,\mathrm{d}p \tag{10}$$

these conditions can be described in simpler forms;

$$\tilde{a}(s, 0; \mathcal{Q}, \mathcal{P}) = \delta(\mathcal{Q})\delta(\mathcal{P} - s) \tag{11}$$

$$\tilde{a}(0, t; \mathcal{Q}, \mathcal{P}) = \delta(\mathcal{Q} - t)\delta(\mathcal{P}) \tag{12}$$

$$\tilde{a}(s, t; \mathcal{Q}, \mathcal{P})^{*} = \tilde{a}(-s, -t; -\mathcal{Q}, -\mathcal{P})\,\mathrm{e}^{-\mathrm{i}\mathcal{Q}\mathcal{P}/\hbar} \tag{13}$$

$$\int_{-\infty}^{\infty} \mathrm{d}s\,\mathrm{d}t\,\tilde{a}(s, t; \mathcal{Q}', \mathcal{P}')^{*}\tilde{a}(s, t; \mathcal{Q}, \mathcal{P}) = \delta(\mathcal{Q} - \mathcal{Q}')\delta(\mathcal{P} - \mathcal{P}'). \tag{14}$$

2.1. New conditions arising from translation and parity transformation

For the classical theory, the distribution function $\rho(q, p)$ on phase space is transformed in the following way in a Galilean system;

$$\rho'(q', p') = \rho(q, p) = \rho(q' - a, p' - b)$$

under the translation on phase space,

$$q \to q' = q + a \qquad p \to p' = p + b.$$

And under the parity transformation on phase space

$$q \to q' = -q \qquad p \to p' = -p$$

the distribution function is changed into $\rho'(q', p')$

$$\rho'(q', p') = \rho(q, p) = \rho(-q', -p').$$

Thus, in the quantum theory, we hope that the Fano operator is transformed as follows:

$$U_{\text{cont}}(a, b)\hat{\Delta}(q, p)U_{\text{cont}}^{-1}(a, b) = \hat{\Delta}(q - a, p - b) \tag{15}$$

and

$$T_{\text{cont}}\hat{\Delta}(q, p)T_{\text{cont}}^{-1} = \hat{\Delta}(-q, -p). \tag{16}$$

Here, $U_{\text{cont}}(a, b)$ and T_{cont} are the unitary operators which are defined by

$$U_{\text{cont}}(a, b) = \exp\left[\mathrm{i}\frac{\hat{P}a - \hat{Q}b}{\hbar}\right] \tag{17}$$

$$T_{\text{cont}} = \exp\left[-\mathrm{i}\pi\frac{\hat{Q}^2 + \hat{P}^2}{2\hbar}\right]. \tag{18}$$

It is easily shown that these unitary operators $U_{\text{cont}}(a, b)$ and T_{cont} induce the translation and parity transformations, respectively,

$$U_{\text{cont}}(a, b)\hat{Q}U_{\text{cont}}^{-1}(a, b) = \hat{Q} + a \qquad U_{\text{cont}}(a, b)\hat{P}U_{\text{cont}}^{-1}(a, b) = \hat{P} + b \tag{19}$$

$$T_{\text{cont}}\hat{Q}T_{\text{cont}}^{-1} = -\hat{Q} \qquad T_{\text{cont}}\hat{P}T_{\text{cont}}^{-1} = -\hat{P}. \tag{20}$$

For the coefficients in the expansion (9), these conditions (15) and (16) become

$$a(q, p; \mathcal{Q}, \mathcal{P})\,\mathrm{e}^{\mathrm{i}\mathcal{Q}b/\hbar}\,\mathrm{e}^{-\mathrm{i}a\mathcal{P}/\hbar} = a(q - a, p - b; \mathcal{Q}, \mathcal{P}) \tag{21}$$

$$a(q, p; -\mathcal{Q}, -\mathcal{P}) = a(-q, -p; \mathcal{Q}\mathcal{P}). \tag{22}$$

Using the Fourier transformed coefficients $\tilde{a}(s, t; \mathcal{Q}, \mathcal{P})$, these conditions are given by

$$\tilde{a}(s, t; \mathcal{Q}, \mathcal{P}) = \mathrm{e}^{-\mathrm{i}(\mathcal{Q}-t)b/\hbar}\,\mathrm{e}^{\mathrm{i}a(\mathcal{P}-s)/\hbar}\tilde{a}(s, t; \mathcal{Q}, \mathcal{P}) \tag{23}$$

$$\tilde{a}(s, t; -\mathcal{Q}, -\mathcal{P}) = \tilde{a}(-s, -t; \mathcal{Q}, \mathcal{P}). \tag{24}$$

Thus, we have simple equations from assumptions (15) and (16). In the next subsection we try to find the Fano operator satisfying conditions (11)–(14), (23) and (24).

2.2. The Fano operator under the new conditions

From equation (23), we have

$$\tilde{a}(s, t; \mathcal{Q}, \mathcal{P}) = F(s, t)\delta(\mathcal{Q} - t)\delta(\mathcal{P} - s) \tag{25}$$

where $F(s, t)$ is a function of s and t which is determined by other conditions.

Taking account of the condition (24), we obtain the condition for the function $F(s, t)$,

$$F(s, t) = F(-s, -t). \tag{26}$$

From conditions (26) and (13), the function $F(s, t)$ should satisfy

$$F^*(s, t) = \mathrm{e}^{-\mathrm{i}ts/\hbar}F(-s, -t) = \mathrm{e}^{-\mathrm{i}ts/\hbar}F(s, t)$$

so that we get

$$F(s,t) = R(s,t)\,\mathrm{e}^{\mathrm{i}st/2\hbar}$$

where $R(s,t)$ is a real function of s and t. The value of the square of this function $R(s,t)$ is restricted to unity by the condition (14)

$$R^2(s,t) = 1 \qquad \text{or} \qquad R(s,t) = \pm 1.$$

Because of the condition (11), we should choose +1 as $R(s,t)$ and we have unique solution

$$\tilde{a}(s,t;\mathcal{Q},\mathcal{P}) = \mathrm{e}^{-\mathrm{i}\mathcal{QP}/\hbar}\delta(\mathcal{Q}-t)\delta(\mathcal{P}-s).$$

Here we assumed that the function $R(s,t)$ is continuous.

3. The Wigner function on lattice 'phase space'

In this section, we try to determine the Fano operator by a similar method to the one we adopted in the preceding section. For lattice 'phase space' with N^2 sites, we can obtain the conditions corresponding to the conditions (A)–(C) by replacing the integration by summation over Z_N in equations (11)–(14),

$$\sum_{p\in Z_N} \hat{\Delta}(q,p) = |q,\hat{Q}\rangle\langle q,\hat{Q}| \tag{27}$$

$$\sum_{q\in Z_N} \hat{\Delta}(q,p) = |p,\hat{P}\rangle\langle p,\hat{P}| \tag{28}$$

$$\hat{\Delta}^\dagger(q,p) = \hat{\Delta}(q,p) \tag{29}$$

$$\sum_{q,p\in Z_N} (\hat{\Delta}^\dagger)_{q_1q_2}(q,p)(\hat{\Delta})_{q_3q_4}(q,p) = \frac{1}{N}\delta^{(N)}_{q_1q_4}\delta^{(N)}_{q_3q_2} \tag{30}$$

where $|q,\hat{Q}\rangle$ and $|p,\hat{P}\rangle$ are eigenvectors of the 'coordinate' and 'momentum' operators with eigenvalues q $(q\in Z_N)$ and p $(p\in Z_N)$, respectively and $\delta^{(N)}_{q_1q_2}$ is Kronecker's delta on Z_N,

$$\delta^{(N)}_{q_1q_2} = \begin{cases} 1 & (q_1 \bmod N = q_2) \\ 0 & (q_1 \bmod N \neq q_2) \end{cases}.$$

We decompose the Fano operator $\hat{\Delta}(q,p)$ into matrices S^nP^m $(n,m=0,1,2,\ldots,N-1)$,

$$\hat{\Delta}(q,p) = \sum_{n,m\in Z_N} a(q,p;n,m)S^nP^m \tag{31}$$

where the matrices S and P are defined by

$$S = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 \\ 1 & & & 0 \end{pmatrix} \tag{32}$$

$$P = \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & & & \vdots \\ \vdots & & \omega^n & & \vdots \\ \vdots & & & \ddots & 0 \\ 0 & & & 0 & \omega^{(N-1)} \end{pmatrix} \tag{33}$$

and ω is a primitive Nth root of unity,

$$\omega = \mathrm{e}^{\frac{2\pi\mathrm{i}}{N}}.$$

These matrices satisfy commutation relation

$$S^n P^m = \omega^{nm} P^m S^n \qquad (n, m = \text{integer}). \tag{34}$$

This commutation relation appears similar to the commutation relation between operators $\mathrm{e}^{-\mathrm{i}\mathcal{P}\hat{Q}/\hbar}$ and $\mathrm{e}^{\mathrm{i}\mathcal{Q}\hat{P}/\hbar}$ used in the preceding section

$$\mathrm{e}^{\mathrm{i}\mathcal{Q}\hat{P}/\hbar}\,\mathrm{e}^{-\mathrm{i}\mathcal{P}\hat{Q}/\hbar} = \mathrm{e}^{-\mathrm{i}\mathcal{P}\mathcal{Q}/\hbar}\,\mathrm{e}^{-\mathrm{i}\mathcal{P}\hat{Q}/\hbar}\,\mathrm{e}^{\mathrm{i}\mathcal{Q}\hat{P}/\hbar}$$

and we can think that matrices P and S correspond to $\mathrm{e}^{-\mathrm{i}\hat{Q}/\hbar}$ and $\mathrm{e}^{\mathrm{i}\hat{P}/\hbar}$, respectively. It is natural that the eigenvectors $|q, \hat{Q}\rangle$ and $|p, \hat{P}\rangle$ are regarded as eigenvectors with respect to the matrices P and S, respectively:

$$P|q, \hat{Q}\rangle = \omega^q |q, \hat{Q}\rangle \tag{35}$$

$$S|p, \hat{P}\rangle = \omega^{-p} |p, \hat{P}\rangle \tag{36}$$

where the eigenvector $|p, \hat{P}\rangle$ is expressed as a linear combination of the eigenvectors $|q, \hat{Q}\rangle$ by the discrete Fourier transform,

$$|p, \hat{P}\rangle = \frac{1}{\sqrt{N}} \sum_{q\in Z_N} \omega^{-pq} |q, \hat{Q}\rangle.$$

We will use this correspondence of P and S to $\mathrm{e}^{-\mathrm{i}\hat{Q}/\hbar}$ and $\mathrm{e}^{\mathrm{i}\hat{P}/\hbar}$ in order to define the translation and parity transformation in lattice 'phase space'.

Using the coefficients $a(q, p; n, m)$, conditions (27)–(30) become

$$\sum_{p\in Z_N} a(q, p; n, m) = \frac{1}{N}\omega^{-qm}\delta_{n,0} \tag{37}$$

$$\sum_{q\in Z_N} a(q, p; n, m) = \frac{1}{N}\omega^{pn}\delta_{m,0} \tag{38}$$

$$a(q, p; n, m) = \omega^{-nm} a(q, p : N - n, N - m) \tag{39}$$

$$\sum_{q,p\in Z_N} a(q, p; n, m) a(q, p; k, l) = \frac{1}{N^2}\delta^{(N)}_{m,k}\delta^{(N)}_{n,l}. \tag{40}$$

In order to obtain equation (40), we used the relation

$$\delta^{(N)}_{a,a'}\delta^{(N)}_{b,b'} = \frac{1}{N^2}\sum_{n,m\in Z_N} (S^n P^m)_{ab} (S^n P^{-m})_{a'b'}.$$

We introduce the Fourier transformed coefficients $\tilde{a}(s, t; n, m)$

$$\tilde{a}(s, t; n, m) = \frac{1}{N^2}\sum_{q,p\in Z_N} \omega^{qs}\omega^{-pt} a(q, p; n, m). \tag{41}$$

The above conditions (37)–(40) become

$$\tilde{a}(s, 0; n, m) = \frac{1}{N^2}\delta^{(N)}_{n,0}\delta^{(N)}_{m,s} \tag{42}$$

$$\tilde{a}(0, t; n, m) = \frac{1}{N^2}\delta^{(N)}_{n,t}\delta^{(N)}_{m,0} \tag{43}$$

$$\tilde{a}(s, t; n, m) = \omega^{-nm}\tilde{a}^*(-s, -t; -n, -m) \tag{44}$$

$$\sum_{s,t\in Z_N} \tilde{a}(s, t; n, m)^* \tilde{a}(s, t; k, l) = \frac{1}{N^4}\delta^{(N)}_{n,k}\delta^{(N)}_{m,l}. \tag{45}$$

3.1. New conditions arising from translation and parity transformation

As we explained in the preceding subsection, since we consider that the matrices P and S as $e^{-i\hat{Q}/\hbar}$ and $e^{i\hat{P}/\hbar}$, from the definition (17) of $U_{\text{cont}}(a,b)$, the unitary matrices $U_{\text{dis}}(a,b)$ for the translation in lattice 'phase space' are obtained,

$$U_{\text{cont}}(a,b) = \exp\left[i\frac{\hat{P}a - \hat{Q}b}{\hbar}\right] \to U_{\text{dis}}(a,b) = P^b S^a \tag{46}$$

where we remove the factor $\exp\left[-i\frac{ab}{2\hbar}\right]$ which appears when the operator $U_{\text{cont}}(a,b)$ is divided into two operators;

$$\exp\left[i\frac{\hat{P}a - \hat{Q}b}{\hbar}\right] = \exp\left[-i\frac{ab}{2\hbar}\right]\exp\left[-i\frac{b\hat{Q}}{\hbar}\right]\exp\left[i\frac{a\hat{P}}{\hbar}\right]$$

since this factor is not a function defined on $Z_N \times Z_N$ and it does not influence the following result to remove this factor. From commutation relation (34), we obtain

$$\begin{cases} P \to P' = U_{\text{dis}}(a,b)PU_{\text{dis}}^{-1}(a,b) = \omega^a P \\ S \to S' = U_{\text{dis}}(a,b)SU_{\text{dis}}^{-1}(a,b) = \omega^{-b} S \end{cases}. \tag{47}$$

This transformation is similar to the transformation of $e^{-i\hat{Q}/\hbar}$ and $e^{i\hat{P}/\hbar}$ by unitary operator $U_{\text{cont}}(a,b)$.

Imitating equation (15), we assume that the Fano operator satisfies

$$U_{\text{dis}}(a,b)\hat{\Delta}(q,p)U_{\text{dis}}^{-1}(a,b) = \Delta(q-a,p-b). \tag{48}$$

Similarly, for the parity transformation, we assume as follows

$$T_{\text{dis}}\hat{\Delta}(q,p)T_{\text{dis}}^{-1} = \hat{\Delta}(-q,-p) \tag{49}$$

where T_{dis} is the unitary matrix for the parity transformation, which is given by

$$T_{\text{dis}} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \end{pmatrix} \quad \text{or} \quad (T)_{\alpha,\beta} = \delta^{(N)}_{\alpha,N-\beta}. \tag{50}$$

It is checked that this unitary matrix T_{dis} transforms matrices P and S into inverse matrices P^{-1} and S^{-1}, respectively,

$$\begin{cases} P \to P' = T_{\text{dis}}PT_{\text{dis}}^{-1} = P^{-1} \\ S \to S' = T_{\text{dis}}ST_{\text{dis}}^{-1} = S^{-1} \end{cases}. \tag{51}$$

For the coefficients $a(q,p;n,m)$ in the expansion (31), these conditions (48) and (49) become

$$a(q,p;n,m) = \omega^{-ma+nb}a(q-a,p-b;n,m)$$
$$a(q,p;n,m) = a(-q,-p;-n,-m)$$

and the coefficients $\tilde{a}(s,t;n,m)$ in equation (41) satisfy

$$\tilde{a}(s,t;n,m) = \omega^{a(s-m)}\omega^{-b(t-n)}\tilde{a}(s,t;n,m) \tag{52}$$
$$\tilde{a}(s,t;n,m) = \tilde{a}(-s,-t;-n,-m). \tag{53}$$

3.2. The Fano operator under the new conditions

In order that the coefficients $\tilde{a}(s,t;n,m)$ satisfy the condition (52) for arbitrary integers a and b, $\tilde{a}(s,t;n,m)$ should be proportional to $\delta^{(N)}_{t,n}\delta^{(N)}_{s,m}$,

$$\tilde{a}(s,t;n,m) = F(s,t)\delta^{(N)}_{t,n}\delta^{(N)}_{s,m} \tag{54}$$

where $F(s,t)$ is a complex valued function. And, from the condition (53), the function $F(s,t)$ should be a symmetric function under the reflection $s \to -s, t \to -t$

$$F(-s,-t) = F(s,t). \tag{55}$$

Condition (44) with the above equation determines the phase factor up to sign,

$$F(s,t) = \omega^{-st/2}R(s,t) \tag{56}$$

where $R(s,t)$ is a real function. Substituting this equation (56) into the condition (45), we can see that

$$R(s,t)^2 = 1.$$

Thus, we obtain the same condition as we did for a one-dimensional quantum system in the preceding section. In that case, we could determine the $R(s,t)$ since we assumed that it is a continuous function. However, we have many ways of assigning ± 1 to each site on a lattice 'phase space'. For example, in the case where N is odd, if we choose

$$R(s,t) = (-1)^{st} = \omega^{Nst/2}$$

we get the Fano operator which is given by Cohendet *et al* (1988).

4. Summary and discussion

We tried to determine the Fano operator uniquely under the assumptions for translation and parity transformation. For a quantum system confined to a line $\mathbb{R}$, we found only one Fano operator satisfying these conditions and three original conditions. In contrast to this case, for a lattice 'phase space' which includes N^2 sites, we could not determine the Fano operator uniquely.

We considered the map from a point on phase space to the point rotated about the original point $(q,p) = (0,0)$ by π as the parity transformation. However, there are quantum systems where we had better consider the rotation about another point, instead of the origin, as the parity transformation. For example, in spin systems, the parity transformation is corresponding to exchanging between eigenstates for eigenvalues ω^k and ω^{N-1-k} of matrix S. This transformation is equivalent to the rotation about the point $\left(\frac{N-1}{2}, \frac{N-1}{2}\right)$. The rotation $T_{\text{cont}}(c,d)$ and $T_{\text{dis}}(c,d)$ about the point $(c/2, d/2)$ by π for the phase space for the system confined to a line $\mathbb{R}$ and discrete 'phase space' can be described in terms of combination of translation and the rotation about the origin;

$$(q,p) \overset{U^{-1}(\frac{c}{2},\frac{d}{2})}{\longrightarrow} \left(q-\frac{c}{2}, p-\frac{d}{2}\right) \overset{T}{\longrightarrow} \left(\frac{c}{2}-q, \frac{d}{2}-p\right) \overset{U(\frac{c}{2},\frac{d}{2})}{\longrightarrow} (c-q, d-p) \tag{57}$$

and we have

$$T(c,d) = U(c,d)T. \tag{58}$$

Hereafter, the subscripts 'cont' and 'dis' are dropped, as we have the same equations for both cases. From equation (58), we can expect that the assumptions for this transformation do not give rise to essentially different conditions from the ones we considered in the preceding sections. Indeed, if we assume the behaviour

$$T(c,d)\hat{\Delta}(q,p)T^{-1}(c,d) = \hat{\Delta}(c-q,d-p) \tag{59}$$

we have

$$\tilde{a}(s,t;\mathcal{Q},\mathcal{P}) = \mathrm{e}^{-\mathrm{i}d(\mathcal{Q}-t)/\hbar}\,\mathrm{e}^{\mathrm{i}c(\mathcal{P}-s)/\hbar}\tilde{a}(-s,-t;\mathcal{Q},\mathcal{P}) \tag{60}$$

$$\tilde{a}(s,t;n,m) = \omega^{c(s-m)}\omega^{-d(t-n)}\tilde{a}(-s,-t;n,m). \tag{61}$$

Owing to the delta functions or Kronecker's delta in equations (25) and (54), the factors $\mathrm{e}^{-\mathrm{i}d(\mathcal{Q}-t)/\hbar}\,\mathrm{e}^{\mathrm{i}c(\mathcal{P}-s)/\hbar}$ and $\omega^{c(s-m)}\omega^{-d(t-n)}$ vanish, so that these conditions reduce to the ones (24) and (53) we considered.

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